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Dynamic response of an infinite beam and plate to a stochastic train of moving forces

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Abstract

In this paper, the problem of a dynamic response of an infinite beam and a plate resting on a two-parametric foundation ("Pasternak foundation") to the passage of a train of random forces is studied. This train of forces idealizes the flow of vehicles having random weights and travelling at the same speed. It is assumed that the occurrence process is either a Poisson process or a renewal (Erlang) process. Explicit expressions for the cumulants (semi-invariants) of the beam and plate response are provided in the case of the Poisson process and the expected value and the variance are provided in the case of the renewal process. We present two different situations and solutions: one for the arbitrary locations of the forces on the beam and another when one of the forces is located in the point in which the response of the beam has the maximum value. The first model can be used to estimate the reliability of the beam or the plate with respect to fatigue; the second model can be useful in the reliability problem of the beam or the plate with respect to the maximum response.

1. Introduction

The problem of a dynamic response of a structure subjected to moving loads is interesting and important. Vibrations connected with this issue occur for example in the roadways loaded by traffic. In the past in most studies the moving load has been regarded as deterministic, however in the last years also stochastic approaches have been presented. The problem of vibrations of an infinite beam or a plate resting on an elastic foundation subjected to a moving load is important in dynamics of roadways and railways and has been considered in some papers also in stochastic approaches. Knowles [1] dealt with the problem of an infinitely long beam subjected to a moving concentrated force, the position of which is described by a strictly stationary first order, stationary Gaussian, or Wiener stochastic process. The stochastic analysis of a beam on a random foundation subjected to a moving load has been considered by Fryba et al. [2] and by Andersen and Nielsen [3]. The problem of vibrations and the reliability of a finite beam due to a random train of moving point forces have been analyzed by Tung [4–6], Sniady et al. [7–10], Zibdeh and Rackwitz [11,12], and Riccardi [13]. The vibrations of a suspension bridge under a random train of moving loads have been discussed by Bryja and

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Sniady [14–16]. The problem of vibration of an infinite beam resting on a foundation subjected to a random train of moving point forces constituting a Poisson process has been considered by Sniady et al. [17].

In this paper, the problem of the dynamic response of an infinite beam and a plate resting on a two-parametric foundation ("Pasternak foundation") to the passage of a train of random forces is studied. This train of forces idealizes the flow of vehicles having random weights and travelling at the same speed. It is assumed that the occurrence process is a Poisson process or a renewal (Erlang) process. Explicit expressions for the cumulants (semi-invariants) of the beam and plate response are provided in the case of the Poisson process and the expected value and the variance are provided in the case of the renewal process. We present two different situations and solutions: one for the arbitrary locations of the forces on the beam or the plate and another when one of the forces is located in the point in which the response of the beam or the plate has the maximum value. The first model can be used to estimate the reliability of the beam or the plate with respect to fatigue, the second model can be useful in the reliability problem of the beam or the plate with respect to the maximum response.

2. Formulation of the problem and general solutions for a beam

Consider a beam of an infinite length, laying on a Pasternak, two-parameter subsoil, subjected to a train of forces, all moving in the same direction with constant velocity v (Fig. 1). Vibrations of the beam due to this train of forces are described by the equation:

$$\mathrm{EI}\frac{\partial^4 w(x,t)}{\partial x^4} - k_1\frac{\partial^2 w(x,t)}{\partial x^2} + k_0 w(x,t) + m\frac{\partial^2 w(x,t)}{\partial t^2} = F(x,t), \tag{1}$$

where EI denotes the flexural rigidity of the beam, m denotes the mass of the beam per unit of length, k_0 and k_1 are the elastic and the shear stiffness of the foundation, $w(x,t)$ is the vertical displacement of the beam and $F(x,t)$ denotes the load process.

In the case of a random train of moving forces (vehicles) the loading process has a form:

$$F(x,t) = \sum_{k=-\infty}^{\infty} A_k \delta[x - (x_k + vt)], \tag{2}$$

where δ denotes Dirac delta function, whereas amplitudes A_k are assumed to be random variables which are mutually independent and independent of the instants. It is assumed that the expected value $E[A_k{}^n]$ does not depend on k and is known for each $n = 1, 2, \ldots$.

The inter-arrival times of the moving forces are regarded as random variables and constitute the Poisson process $N(x,t)$ with parameter $\lambda(x,t)$. The function $\lambda(x,t)$ describes the intensity of the load distribution, that is to say the expected value of the vehicle number per unit of length at the x-point and the t-moment. In accordance with the road traffic theory [18,19] $\lambda(x,t)$ must satisfy the equation of continuity shown below, which expresses that the variation of a vehicle number within any range dx and time interval dt must be equal to the difference between the number of the vehicles entering the range dx at the x and leaving it at the point x+dx:

$$\frac{\partial\lambda(x,t)}{\partial t} + v\frac{\partial\lambda(x,t)}{\partial x} = 0. \tag{3}$$

Eq. (3) is satisfied when the $\lambda(x,t)$ function has the following form:

$$\lambda(x,t) = \lambda(x - vt) = \lambda(\xi). \tag{4}$$

In this paper the problem will be confined to the homogeneous case $\lambda(\xi) = \lambda = \text{const}$.

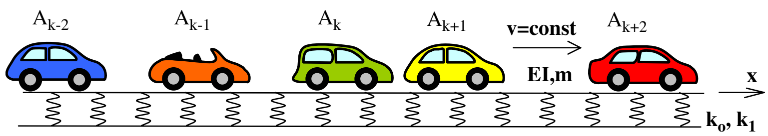

Fig 1. The general plan for a beam.

Let us introduce the moving coordinate system relating the variable t with x by the new coordinate $\xi = x - vt$. Now the issue of the beam dynamics comes down to the steady-state vibrations.

Let $\mathrm{d}N(\xi)$ denote the number of vehicles within the range $(\xi,\ \xi + \mathrm{d}\xi)$ then for the Poisson distribution the following relationships hold true:

$$\begin{aligned} &E\left[\mathrm{d}N^k(\xi)\right] = \lambda\,\mathrm{d}\xi && \text{for } k = 1, 2, \ldots, \\ &E[\mathrm{d}N(\xi_1)\,\mathrm{d}N(\xi_2)] = \lambda^2\,\mathrm{d}\xi_1\,\mathrm{d}\xi_2 && \text{for } \xi_1 \neq \xi_2. \end{aligned} \tag{5}$$

There are two important setups we shall consider in the following. In the first case the deflection is counted at the arbitrarily chosen ξ-section and none of the vehicle locations is known, in the second one the ξ-section is located at the place in which the deflection of the beam is maximal which means that for the steady-state vibrations it is considered just under one of the forces (Fig. 2).

Let the function $H(\xi,\ \xi_0)$ be the dynamic influence function, namely it is equal to the steady-state dislocation of a beam at a ξ-point caused by a one singular point load moving with a constant velocity at a ξ_0-point.

According to those facts the deflection function for the first and the second case fulfils:

$$w_{\mathrm{I}}^P(\xi) = \int_{-\infty}^{\infty} A(\xi_0)H(\xi, \xi_0)\,\mathrm{d}N(\xi_0), \tag{6}$$

$$w_{\mathrm{II}}^P(\xi) = A(\xi)H(\xi, \xi) + \int_{-\infty}^{\infty} A(\xi_0)H(\xi, \xi_0)\,\mathrm{d}N(\xi_0). \tag{7}$$

The expected value of the random functions $w_{\mathrm{I}}^P(\xi)$ and $w_{\mathrm{II}}^P(\xi)$ amounts to

$$E\left[w_{\mathrm{I}}^P(\xi)\right] = E[A]\lambda \int_{-\infty}^{\infty} H(\xi, \xi_0)\,\mathrm{d}\xi_0, \tag{8}$$

$$E\left[w_{\mathrm{II}}^P(\xi)\right] = E[A]H(\xi, \xi) + E[A]\lambda \int_{-\infty}^{\infty} H(\xi, \xi_0)\,\mathrm{d}\xi_0, \tag{9}$$

whereas the variance

$$\mathrm{Var}\left[w_{\mathrm{I}}^P(\xi)\right] = E\left[A^2\right]\lambda \int_{-\infty}^{\infty} H^2(\xi, \xi_0)\,\mathrm{d}\xi_0, \tag{10}$$

$$\begin{aligned} \mathrm{Var}\left[w_{\mathrm{II}}^P(\xi)\right] = {} & \mathrm{Var}[A]H^2(\xi, \xi) \\ & + E\left[A^2\right]\lambda \int_{-\infty}^{\infty} H^2(\xi, \xi_0)\,\mathrm{d}\xi_0. \end{aligned} \tag{11}$$

The expected value and the variance are the cumulants (semi-invariants) of the first and the second order, respectively. In a general case the nth order cumulant of the $w_{\mathrm{I}}^P(\xi)$ and the $w_{\mathrm{II}}^P(\xi)$ can be written, accordingly:

$$\kappa_n\left[w_{\mathrm{I}}^P(\xi)\right] = E[A^n]\lambda \int_{-\infty}^{\infty} H^n(\xi, \xi_0)\,\mathrm{d}\xi_0, \tag{12}$$

$$\begin{aligned} \kappa_n\left[w_{\mathrm{II}}^P(\xi)\right] = {} & \kappa_n[A]H^n(\xi, \xi) \\ & + E[A^n]\lambda \int_{-\infty}^{\infty} H^n(\xi, \xi_0)\,\mathrm{d}\xi_0, \end{aligned} \tag{13}$$

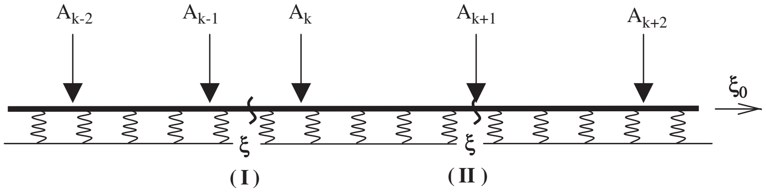

Fig. 2. The problem of the first (I) and the second case (II)—the simplified scheme for a beam.

where κ_n[A] denotes the nth order cumulant of the random value A_i, the I and the II subscript refers to the first and the second case whereas the P superscript indicates that the Poisson process is considered. The proof of the expression from Eq. (13) is given in Appendix A.

The above formulas can be simply modified when the traffic structure is taken into account. Additionally, in the second case if one assumes that the vehicle in the ξ-section belongs to the heaviest class, then we get the following formulas for:

- Random functions of deflection:

$$w_{\mathrm{I}}^P(\xi) = \sum_r \int_{-\infty}^{\infty} A_r(\xi_0) H_r(\xi, \xi_0)\, \mathrm{d}N_r(\xi_0), \tag{14}$$

$$\begin{aligned} w_{\mathrm{II}}^P(\xi) = {} & A_{\max}(\xi) H_{\max}(\xi, \xi) \\ & + \sum_r \int_{-\infty}^{\infty} A_r(\xi_0) H_r(\xi, \xi_0)\, \mathrm{d}N_r(\xi_0). \end{aligned} \tag{15}$$

- Expected values:

$$E\left[w_{\mathrm{I}}^P(\xi)\right] = \sum_r E[A_r]\lambda_r \int_{-\infty}^{\infty} H_r(\xi, \xi_0)\, \mathrm{d}\xi_0, \tag{16}$$

$$\begin{aligned} E\left[w_{\mathrm{II}}^P(\xi)\right] = {} & E[A_{\max}] H_{\max}(\xi, \xi) \\ & + \sum_r E[A_r]\lambda_r \int_{-\infty}^{\infty} H_r(\xi, \xi_0)\, \mathrm{d}\xi_0. \end{aligned} \tag{17}$$

- Variances:

$$\mathrm{Var}\left[w_{\mathrm{I}}^P(\xi)\right] = \sum_r E\left[A_r^2\right]\lambda_r \int_{-\infty}^{\infty} H_r^2(\xi, \xi_0)\, \mathrm{d}\xi_0, \tag{18}$$

$$\begin{aligned} \mathrm{Var}\left[w_{\mathrm{II}}^P(\xi)\right] = {} & \mathrm{Var}[A_{\max}] H_{\max}^2(\xi, \xi) \\ & + \sum_r E\left[A_r^2\right]\lambda_r \int_{-\infty}^{\infty} H_r^2(\xi, \xi_0)\, \mathrm{d}\xi_0. \end{aligned} \tag{19}$$

- nth order cumulants:

$$\kappa_n\left[w_{\mathrm{I}}^P(\xi)\right] = \sum_r E\left[A_r^n\right]\lambda_r \int_{-\infty}^{\infty} H_r^n(\xi, \xi_0)\, \mathrm{d}\xi_0, \tag{20}$$

$$\begin{aligned} \kappa_n\left[w_{\mathrm{II}}^P(\xi)\right] = {} & \kappa_n[A_{\max}] H_{\max}^n(\xi, \xi) \\ & + \sum_r E\left[A_r^n\right]\lambda_r \int_{-\infty}^{\infty} H_r^n(\xi, \xi_0)\, \mathrm{d}\xi_0, \end{aligned} \tag{21}$$

where the r-iterator refers to the vehicle class, whereas the max denotation is associated with the quantities regarding the class of the heaviest vehicles.

The Poisson process, with an exponential density function for the intervals between consecutive forces, has the property that higher probability is assigned to smaller intervals than to larger intervals. In the case of traffic flow the probability density function for the intervals between consecutive forces approaches zero when the intervals tend to zero [18,19]. This is the property of the Erlang (Pearson type III) process the probability density function of which for the intervals is given by

$$f(\Delta\xi) = \frac{\lambda(\lambda\Delta\xi)^{k-1}}{(k-1)!}\mathrm{e}^{-\lambda\Delta\xi} \tag{22}$$

in which λ is a positive constant and k is integer ($k = 1,2,\ldots$)

The interval between every kth force in the Poisson process constitutes the Erlang process given by expression in Eq. (22). This fact can be used to construct the solution for the response of the beam loaded by a random train of moving forces in the case it is an Erlang process with integer $k = 2$ (see Ref. [20]). Let $N(\xi)$ denote the Poisson process with parameter $\lambda = 2\mu$ where the parameter μ is the intensity of the flow.

As for the Erlang process it is convenient to consider only the second case the deflection in which has the following form:

$$\begin{aligned} w_{\mathrm{II}}^{E}(\xi) &= A(\xi)H(\xi,\xi) \\ &\quad + 0.5\int_{-\infty}^{\infty} A(\xi_0)\left[1-(-1)^{N(\xi_0)}\right]H(\xi,\xi_0)\,\mathrm{d}N(\xi_0). \end{aligned} \tag{23}$$

As in the case of Poisson process the explicit expressions for the expected value and the variance of the random variable $w_{\mathrm{II}}^{E}(\xi)$ are given:

$$\begin{aligned} E\left[w_{\mathrm{II}}^{E}(\xi)\right] &= E[A]H(\xi,\xi) \\ &\quad + E[A]\mu\int_{-\infty}^{\infty}\left[1-\mathrm{e}^{-4\mu|\xi_0|}\right]H(\xi,\xi_0)\,\mathrm{d}\xi_0, \end{aligned} \tag{24}$$

$$\begin{aligned} \mathrm{Var}\left[w_{\mathrm{II}}^{E}(\xi)\right] &= \mathrm{Var}[A]H^2(\xi,\xi) + E\left[A^2\right]\mu\int_{-\infty}^{\infty}\left[1-\mathrm{e}^{-4\mu|\xi_0|}\right]H^2(\xi,\xi_0)\,\mathrm{d}\xi_0 \\ &\quad + 4E^2[A]\mu^2\int_{-\infty}^{\infty}\int_{-\infty}^{\xi_1}\mathrm{e}^{-4\mu|\xi_1|}H(\xi,\xi_1)H(\xi,\xi_2)\,\mathrm{d}\xi_1\,\mathrm{d}\xi_2 \\ &\quad + 4E^2[A]\mu^2\int_{-\infty}^{\infty}\int_{-\infty}^{\xi_1}\mathrm{e}^{-4\mu(|\xi_1|+|\xi 2|)}H(\xi,\xi_1)H(\xi,\xi_2)\,\mathrm{d}\xi_1\,\mathrm{d}\xi_2 \\ &\quad - E^2[A]\mu^2\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\mathrm{e}^{-4\mu|\xi_1-\xi_2|}H(\xi,\xi_1)H(\xi,\xi_2)\,\mathrm{d}\xi_1\,\mathrm{d}\xi_2. \end{aligned} \tag{25}$$

The following relationships were used in order to derive above formulas [20]:

$$\begin{aligned} E\left[\mathrm{d}N^k(\xi)\right] &= \lambda\,\mathrm{d}\xi = 2\mu\,\mathrm{d}\xi \quad \text{for} \quad k = 1,2\ldots, \\ E[\mathrm{d}N(\xi_1)\,\mathrm{d}N(\xi_2)] &= \lambda^2\,\mathrm{d}\xi_1\,\mathrm{d}\xi_2 = 4\mu^2\,\mathrm{d}\xi_1\,\mathrm{d}\xi_2 \quad \text{for} \quad \xi_1 \neq \xi_2, \\ E\left[(-1)^{N(\xi)}\right] &= \mathrm{e}^{-2\lambda\xi} = \mathrm{e}^{-4\mu\xi}. \end{aligned} \tag{26}$$

As one can see the formula for the variance of the deflection is much more complicated in the case of the Erlang process than in the Poisson one. Therefore the problem of the dynamic response will be considered only within the range of the first two cumulants as far as the Erlang distribution is concerned.

After taking into account the traffic structure one gets the expressions as follows:

- Random function of deflection:

$$\begin{aligned} w_{\mathrm{II}}^{E}(\xi) &= A_{\max}(\xi)H_{\max}(\xi,\xi) \\ &\quad + 0.5\sum_r\int_{-\infty}^{\infty} A_r(\xi_0)\left[1-(-1)^{N_r(\xi_0)}\right]H_r(\xi,\xi_0)\,\mathrm{d}N_r(\xi_0). \end{aligned} \tag{27}$$

- Expected value:

$$E\left[w_{\mathrm{II}}^E(\xi)\right] = E[A_{\max}]H_{\max}(\xi,\xi) + \sum_r E[A_r]\mu_r \int_{-\infty}^{\infty}\left[1-\mathrm{e}^{-4\mu_r|\xi_0|}\right]H_r(\xi,\xi_0)\,\mathrm{d}\xi_0. \tag{28}$$

- Variance:

$$\begin{aligned}\mathrm{Var}\left[w_{\mathrm{II}}^E(\xi)\right] &= \mathrm{Var}[A_{\max}]H_{\max}^2(\xi,\xi) + \sum_r E\left[A_r^2\right]\mu_r \int_{-\infty}^{\infty}\left[1-\mathrm{e}^{-4\mu_r|\xi_0|}\right]H_r^2(\xi,\xi_0)\,\mathrm{d}\xi_0 \\ &+ 4\sum_r E^2[A_r]\mu_r^2 \int_{-\infty}^{\infty}\int_{-\infty}^{\xi_1} \mathrm{e}^{-4\mu_r|\xi_1|}H_r(\xi,\xi_1)H_r(\xi,\xi_2)\,\mathrm{d}\xi_1\,\mathrm{d}\xi_2 \\ &- \sum_r E^2[A_r]\mu_r^2 \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \mathrm{e}^{-4\mu_r(|\xi_1|+|\xi_2|)}H_r(\xi,\xi_1)H_r(\xi,\xi_2)\,\mathrm{d}\xi_1\,\mathrm{d}\xi_2 \\ &- \sum_r E^2[A_r]\mu_r^2 \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \mathrm{e}^{-4\mu_r|\xi_1-\xi_2|}H_r(\xi,\xi_1)H_r(\xi,\xi_2)\,\mathrm{d}\xi_1\,\mathrm{d}\xi_2.\end{aligned} \tag{29}$$

One gets the dynamic influence function from the following equation which is obtained from Eq. (1) after introducing the ξ-coordinate and taking Dirac delta into account as a load function:

$$\mathrm{EI}\frac{\partial^4 H(\xi,\xi_0)}{\partial\xi^4} + \left(mv^2-k_1\right)\frac{\partial^2 H(\xi,\xi_0)}{\partial\xi^2} + k_0 H(\xi,\xi_0) = \delta(\xi-\xi_0). \tag{30}$$

The solution for the beam described by Eq. (30) has the form (see Appendix B):

$$H(\xi,\xi_0) = \begin{cases} \dfrac{1}{4\sqrt{\mathrm{EI}\,k_0}}\mathrm{e}^{\alpha(\xi_0-\xi)}\left[\dfrac{\cos\delta(\xi_0-\xi)}{\alpha} - \dfrac{\sin\delta(\xi_0-\xi)}{\delta}\right] & \text{for } \xi \geq \xi_0, \\ \dfrac{1}{4\sqrt{\mathrm{EI}\,k_0}}\mathrm{e}^{-\alpha(\xi_0-\xi)}\left[\dfrac{\cos\delta(\xi_0-\xi)}{\alpha} + \dfrac{\sin\delta(\xi_0-\xi)}{\delta}\right] & \text{for } \xi \leq \xi_0, \end{cases} \tag{31}$$

where

$$\alpha = \frac{1}{2}\sqrt{2\sqrt{\frac{k_0}{\mathrm{EI}}} - \frac{mv^2-k_1}{\mathrm{EI}}} = \frac{a_1}{2}\sqrt{1-\left(\frac{v}{v_{\mathrm{cr}}}\right)^2}, \quad \delta = \frac{1}{2}\sqrt{2\sqrt{\frac{k_0}{\mathrm{EI}}} + \frac{mv^2-k_1}{\mathrm{EI}}} = \frac{a_2}{2}\sqrt{1+\frac{v^2}{v_{\mathrm{cr}}^2-2k_1/m}}$$

and

$$a_1 = \sqrt{2\sqrt{\frac{k_0}{\mathrm{EI}}} + \frac{k_1}{\mathrm{EI}}}, \quad a_2 = \sqrt{2\sqrt{\frac{k_0}{\mathrm{EI}}} - \frac{k_1}{\mathrm{EI}}}, \quad v_{\mathrm{cr}} = \sqrt{\frac{\sqrt{4k_0\,\mathrm{EI}}+k_1}{m}}.$$

Let us see that we can build such a dynamic influence function for a singular vehicle modelled as a system of point loads the reciprocal locations and values of which follow from the wheel space and mass distribution on axles, respectively. Then vehicle dynamic influence function can be described as follows:

$$H_r(\xi,\xi_0) = \sum_{n\text{-th axle}} p_n H(\xi,\xi_0 \pm \Delta\xi_n), \tag{32}$$

where ξ_0 denotes the coordinate of the privileged axle and defines the location of the vehicle on the beam, $\Delta\xi_n$ distance of a respective nth axle to the privileged one, whereas p_n is a nth axle load to vehicle weight ratio.

For particular classes of vehicle [21] the function has the form:

a) cars:

$$H_1(\xi,\xi_0) = 0.5H(\xi,\xi_0+2.5) + 0.5H(\xi,\xi_0).$$

b) small trucks:

$$H_2(\xi,\xi_0) = 0.6H(\xi,\xi_0) + 0.4H(\xi,\xi_0-2.8).$$

c) 2-axle large trucks:

$$H_3(\xi,\xi_0) = 0.35H(\xi,\xi_0+4.3) + 0.65H(\xi,\xi_0),$$

d) 3-axle trucks:

$$H_4(\xi,\xi_0) = 0.32H(\xi,\xi_0+1.3) + 0.42H(\xi,\xi_0) + 0.26H(\xi,\xi_0-3.3),$$

e) 4-axle trucks (trailer trucks):

$$H_5(\xi,\xi_0) = 0.25H(\xi,\xi_0+1.3) + 0.25H(\xi,\xi_0) + 0.32H(\xi,\xi_0-5.5) + 0.18H(\xi,\xi_0-8.8).$$

Graphs of the dynamic influence functions for specified vehicle classes are introduced in Fig. 3.

3. Formulation of the problem and general solutions for a plate

Now consider a plate of an infinite length and finite wideness, simply supported at the sidelong edges, laying on a Pasternak, two-parameter subsoil and subjected to a train of forces moving in the same direction, all with

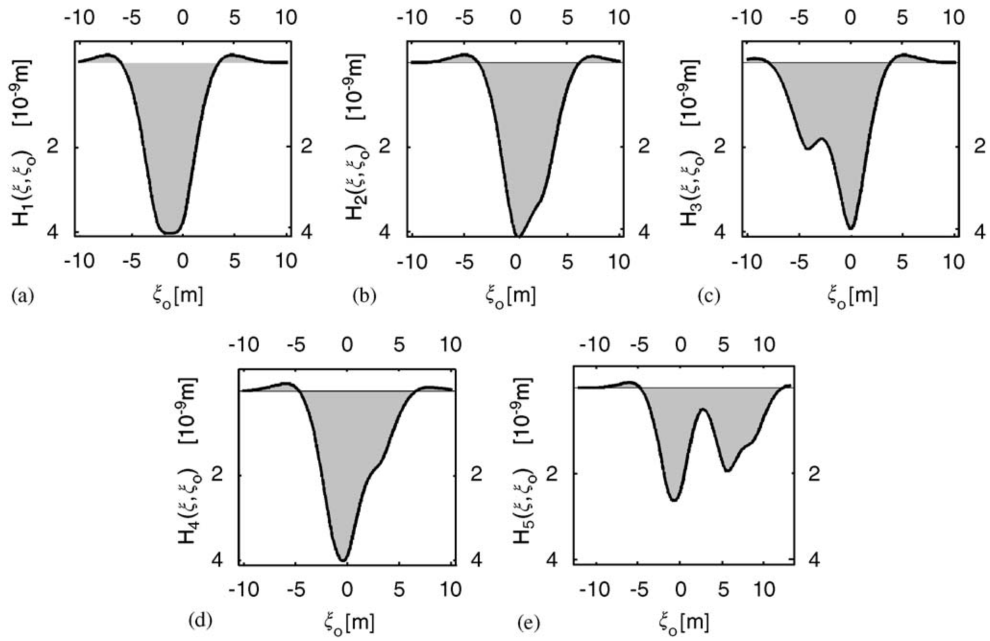

Fig. 3. Dynamic influence functions for specified vehicle classes: (a) cars, (b) small trucks, (c) 2-axle large trucks, (d) 3-axle trucks and (e) 4-axle trucks.

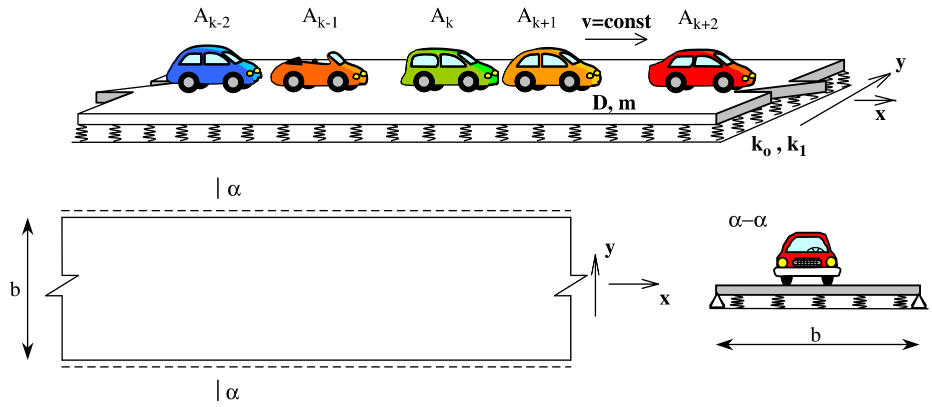

Fig. 4. The general plan for a plate.

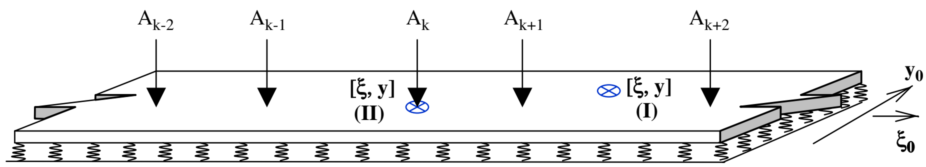

Fig. 5. The problem of the first (I) and the second case (II)—the simplified scheme for a plate.

constant velocity v (Fig. 4). Vibrations of the plate due to this train of forces are described by the equation:

$$D\nabla^2\nabla^2 w(x,y,t) - k_1\nabla^2 w(x,y,t) + k_0 w(x,y,t) + m\frac{\partial^2 w(x,y,t)}{\partial t^2} = F(x,y,t), \tag{33}$$

where D denotes the plate stiffness, the differential operator ∇^2 is the Laplacian operator and all other parameters and functions have the analogous meanings as in the case of a beam.

In the case of a random train of moving forces the loading process has a form:

$$F(x,y,t) = \sum_{k=-\infty}^{\infty} A_k\delta[x-(x_k+vt)]\delta\left[y-y_k\right]. \tag{34}$$

All assumptions for the load process from the previous chapter are still being held. Additionally it is assumed that all vehicles move on the same line, e.g. each vehicle has the same deterministic y-coordinate $y_k = y_0$.

Let $H(\xi, \xi_0, y, y_0)$ be the dynamic influence function which denotes the steady-state dislocation of a plate at a (ξ, y)-point caused by a singular point load moving with a constant velocity at a (ξ_0, y_0)-point.

If the force distances constitute a Poisson distribution then after introducing the variable ξ instead of the x and t one gets the following expressions for the first and the second case (Fig. 5), respectively:

- Random functions of deflection:

$$w_{\mathrm{I}}^P(\xi,y,y_0) = \int_{-\infty}^{\infty} A(\xi_0)H(\xi,\xi_0,y,y_0)\,\mathrm{d}N(\xi_0), \tag{35}$$

$$w_{\mathrm{II}}^P(\xi,y,y_0) = A(\xi)H(\xi,\xi,y,y_0) + \int_{-\infty}^{\infty} A(\xi_0)H(\xi,\xi_0,y,y_0)\,\mathrm{d}N(\xi_0). \tag{36}$$

- nth order cumulants:

$$\kappa_n\left[w_{\rm I}^P(\xi,y,y_0)\right] = E[A^n]\lambda\int_{-\infty}^{\infty} H^n(\xi,\xi_0,y,y_0)\,{\rm d}\xi_0. \tag{37}$$

$$\kappa_n\left[w_{\rm II}^P(\xi,y,y_0)\right] = \kappa_n[A]H^n(\xi,\xi,y,y_0) + E[A^n]\lambda\int_{-\infty}^{\infty} H^n(\xi,\xi_0,y,y_0)\,{\rm d}\xi_0. \tag{38}$$

After taking into account the traffic structure the above expressions change into

$$w_{\rm I}^P(\xi,y,y_0) = \sum_r \int_{-\infty}^{\infty} A_r(\xi_0)H_r(\xi,\xi_0,y,y_0)\,{\rm d}N_r(\xi_0), \tag{39}$$

$$w_{\rm II}^P(\xi,y,y_0) = A_{\max}(\xi)H_{\max}(\xi,\xi,y,y_0) + \sum_r \int_{-\infty}^{\infty} A_r(\xi_0)H_r(\xi,\xi_0,y,y_0)\,{\rm d}N_r(\xi_0) \tag{40}$$

for the random function of deflection, and into

$$\kappa_n\left[w_{\rm I}^P(\xi,y,y_0)\right] = \sum_r E\left[A_r^n\right]\lambda_r\int_{-\infty}^{\infty} H_r^n(\xi,\xi_0,y,y_0)\,{\rm d}\xi_0, \tag{41}$$

$$\kappa_n\left[w_{\rm II}^P(\xi,y,y_0)\right] = \kappa_n[A_{\max}]H_{\max}^n(\xi,\xi,y,y_0) + \sum_r E\left[A_r^n\right]\lambda_r\int_{-\infty}^{\infty} H_r^n(\xi,\xi_0,y,y_0)\,{\rm d}\xi_0 \tag{42}$$

for the nth order cumulant, where the r-iterator and the max denotation have the same meanings as previously.

The next step is to consider the second-order Erlang process. One gets expressions for the following quantities when the second case is taken into account:

- Random function of deflection:

$$w_{\rm II}^E(\xi,y,y_0) = A(\xi)H(\xi,\xi,y,y_0) + 0.5\int_{-\infty}^{\infty} A(\xi_0)\left[1-(-1)^{N(\xi_0)}\right]H(\xi,\xi_0,y,y_0)\,{\rm d}N(\xi_0). \tag{43}$$

- Expected value:

$$E\left[w_{\rm II}^E(\xi,y,y_0)\right] = E[A]H(\xi,\xi,y,y_0) + E[A]\mu\int_{-\infty}^{\infty}\left[1-{\rm e}^{-4\mu|\xi_0|}\right]H(\xi,\xi_0,y,y_0)\,{\rm d}\xi_0. \tag{44}$$

- Variance:

$$\begin{aligned}{\rm Var}\left[w_{\rm II}^E(\xi,y,y_0)\right] &= {\rm Var}\,[A]H^2(\xi,\xi,y,y_0) \\ &+ E\left[A^2\right]\mu\int_{-\infty}^{\infty}\left[1-{\rm e}^{-4\mu|\xi_0|}\right]H^2(\xi,\xi_0,y,y_0)\,{\rm d}\xi_0 \\ &+ 4E^2[A]\mu^2\int_{-\infty}^{\infty}\int_{-\infty}^{\xi_1}{\rm e}^{-4\mu|\xi_1|}H(\xi,\xi_1,y,y_0)H(\xi,\xi_2,y,y_0)\,{\rm d}\xi_1\,{\rm d}\xi_2\end{aligned}$$

$$
\begin{aligned}
&- E^2[A]\mu^2 \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \mathrm{e}^{-4\mu(|\xi_1|+|\xi_2|)} H(\xi,\xi_1,y,y_0)H(\xi,\xi_2,y,y_0)\,\mathrm{d}\xi_1\,\mathrm{d}\xi_2 \\
&- E^2[A]\mu^2 \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \mathrm{e}^{-4\mu|\xi_1-\xi_2|} H(\xi,\xi_1,y,y_0)H(\xi,\xi_2,y,y_0)\,\mathrm{d}\xi_1\,\mathrm{d}\xi_2.
\end{aligned} \tag{45}
$$

One gets analogical expressions if the traffic structure is considered:

$$
\begin{aligned}
w_{\mathrm{II}}^E(\xi,y,y_0) = {} & A_{\max}(\xi)H_{\max}(\xi,\xi,y,y_0) \\
& + 0.5\sum_r \int_{-\infty}^{\infty} A_r(\xi_0)\left[1-(-1)^{N_r(\xi_0)}\right] H_r(\xi,\xi_0,y,y_0)\,\mathrm{d}N_r(\xi_0)
\end{aligned} \tag{46}
$$

for the random function of deflection, and

$$
\begin{aligned}
E\left[w_{\mathrm{II}}^E(\xi,y,y_0)\right] = {} & E[A_{\max}]H_{\max}(\xi,\xi,y,y_0) \\
& + \sum_r E[A_r]\mu_r \int_{-\infty}^{\infty}\left[1-\mathrm{e}^{-4\mu_r|\xi_0|}\right] H_r(\xi,\xi_0,y,y_0)\,\mathrm{d}\xi_0
\end{aligned} \tag{47}
$$

for the expected value of deflection, whereas

$$
\begin{aligned}
\mathrm{Var}\left[w_{\mathrm{II}}^E(\xi,y,y_0)\right] = {} & \mathrm{Var}[A_{\max}]H_{\max}^2(\xi,\xi,y,y_0) \\
& + \sum_r E\left[A_r^2\right]\mu_r \int_{-\infty}^{\infty}\left[1-\mathrm{e}^{-4\mu_r|\xi_0|}\right] H_r^2(\xi,\xi_0,y,y_0)\mathrm{d}\xi_0 \\
& + 4\sum_r E^2\left[A_r^2\right]\mu_r^2 \int_{-\infty}^{\infty}\int_{-\infty}^{\xi_1} \mathrm{e}^{-4\mu_r|\xi_1|} H_r(\xi,\xi_1,y,y_0)H_r(\xi,\xi_2,y,y_0)\,\mathrm{d}\xi_1\,\mathrm{d}\xi_2 \\
& - \sum_r E^2[A_r]\mu_r^2 \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \mathrm{e}^{-4\mu_r(|\xi_1|+|\xi_2|)} H_r(\xi,\xi_1,y,y_0)H_r(\xi,\xi_2,y,y_0)\,\mathrm{d}\xi_1\,\mathrm{d}\xi_2 \\
& - \sum_r E^2[A_r]\mu_r^2 \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \mathrm{e}^{-4\mu_r(|\xi_1-\xi_2|)} H_r(\xi,\xi_1,y,y_0)H_r(\xi,\xi_2,y,y_0)\,\mathrm{d}\xi_1\,\mathrm{d}\xi_2
\end{aligned} \tag{48}
$$

for the variance of deflection.

The dynamic influence function for the plate described by Eq. (33) with the boundary conditions shown in Fig. 4 can be expressed as a single series after using eigenfunctions for approximation in y-direction:

$$
H(\xi,\xi_0,y,y_0) = \sum_n^{\infty} h_n(\xi,\xi_0,y_0)\sin\left(n\pi\frac{y}{b}\right). \tag{49}
$$

Function $h_n(\xi,\ \xi_0,\ y_0)$ can be obtained from the equation:

$$
\begin{aligned}
& D\frac{\partial^4 h_n(\xi,\xi_0,y_0)}{\partial\xi^4} + \left[mv^2 - k_1 - 2D\left(\frac{n\pi}{b}\right)^2\right]\frac{\partial^2 h_n(\xi,\xi_0,y_0)}{\partial\xi^2} \\
& + \left[k_0 + k_1 + \left(\frac{n\pi}{b}\right)^2 + D\left(\frac{n\pi}{b}\right)^4\right]h_n(\xi,\xi_0,y_0) \\
& = \frac{2}{b}\sin\left(n\pi\frac{y_0}{b}\right)\delta(\xi-\xi_0)
\end{aligned} \tag{50}
$$

and has the form:

$$
h_n(\xi,\xi_0,y_0) = \begin{cases}
\dfrac{\mathrm{e}^{\alpha_n(\xi_0-\xi)}}{2Db(\alpha_n^2+\delta_n^2)}\left(\dfrac{\cos\delta_n(\xi_0-\xi)}{\alpha_n} - \dfrac{\sin\delta_n(\xi_0-\xi)}{\delta_n}\right)\sin\left(n\pi\dfrac{y_0}{b}\right) & \text{for } \xi \geqslant \xi_0, \\
\dfrac{\mathrm{e}^{-\alpha_n(\xi_0-\xi)}}{2Db(\alpha_n^2+\delta_n^2)}\left(\dfrac{\cos\delta_n(\xi_0-\xi)}{\alpha_n} + \dfrac{\sin\delta_n(\xi_0-\xi)}{\delta_n}\right)\sin\left(n\pi\dfrac{y_0}{b}\right) & \text{for } \xi \leqslant \xi_0,
\end{cases} \tag{51}
$$

where

$$\alpha_n = \frac{1}{2}\sqrt{2\sqrt{\frac{k_0}{D} + \frac{k_1}{D}\left(\frac{n\pi}{b}\right)^2 + \left(\frac{n\pi}{b}\right)^4} - \frac{mv^2 - k_1}{D} + 2\left(\frac{n\pi}{b}\right)^2},$$

$$\delta_n = \frac{1}{2}\sqrt{2\sqrt{\frac{k_0}{D} + \frac{k_1}{D}\left(\frac{n\pi}{b}\right)^2 + \left(\frac{n\pi}{b}\right)^4} + \frac{mv^2 - k_1}{D} - 2\left(\frac{n\pi}{b}\right)^2}.$$

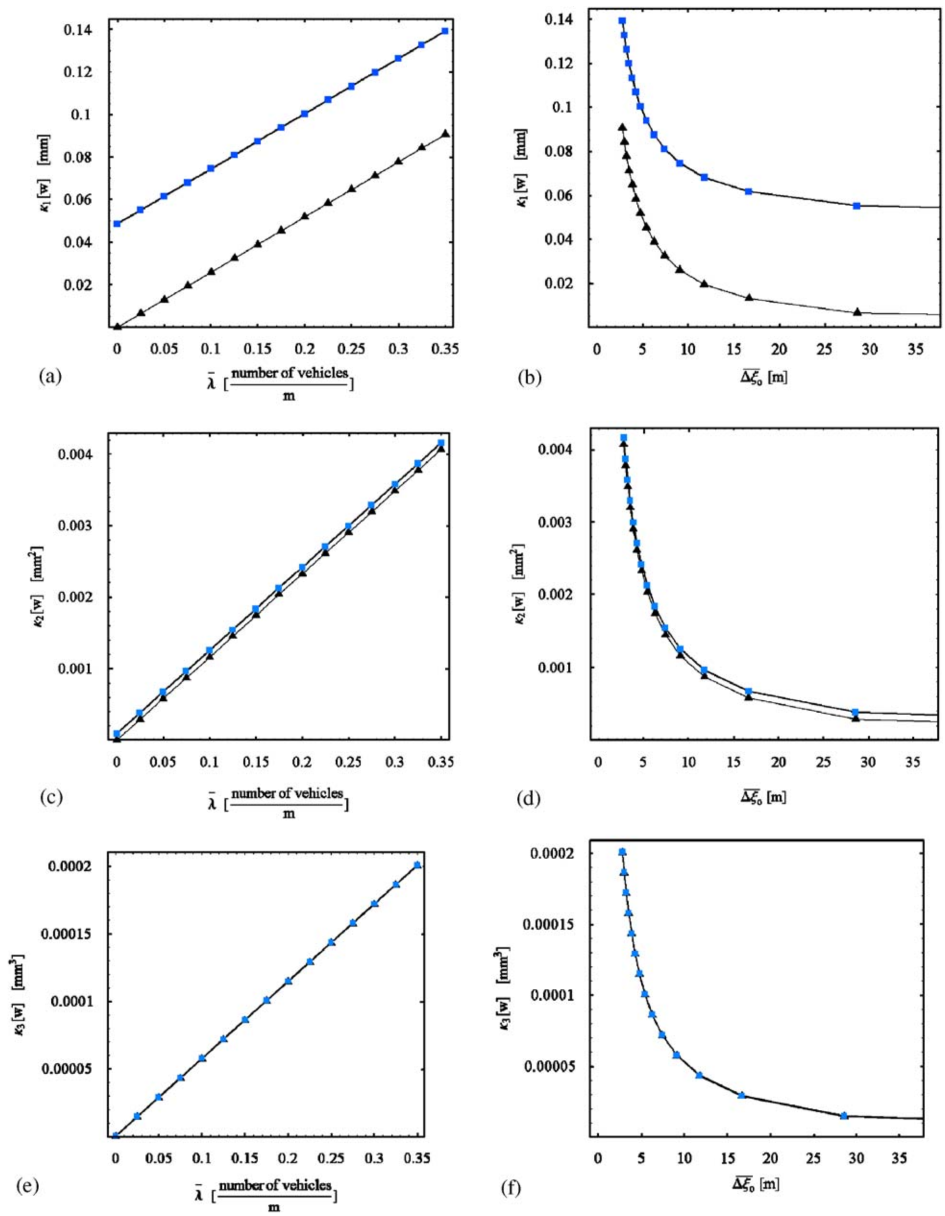

Fig. 6. The cumulants of a beam displacement in the first (I) and the second case (II) for the Poisson process in relation to the average flow intensity λ or the average vehicle distance $\Delta\xi_0$. (a), (b) cumulant of the first order, (c), (d) cumulant of the second order, (e), (f) cumulant of the third order. —▲—, values of cumulants for the first case (I), —■—, values of cumulants for the second case (II).

4. Numerical example and conclusions

In the paper the explicit formulas for the expected value, variance and *n*th order cumulant for the displacement of an infinite beam subjected to a train of random point loads (vehicles) the mutual distances of which constitute a stochastic process were derived.

Figs. 6 and 7 show the example graphs of the cumulants for the beam deflection as a result of numerical calculations in which the following values were used: $m = 2250\,\mathrm{kg\,m^{-1}}$, $\mathrm{EI} = 9 \times 10^7\,\mathrm{N\,m^2}$, $k_1 = 5 \times 10^5\,\mathrm{Nm\,m^{-1}}$, $k_0 = 5 \times 10^7\,\mathrm{N\,m^{-2}}$, $v = 20\,\mathrm{m\,s^{-1}}$. It was assumed that the vehicles weight had a lognormal distribution with the expected value and the standard deviation equal to $m_A = 13\,\mathrm{kN}$ and $\sigma_A = 2.6\,\mathrm{kN}$, respectively. Only one class of vehicles was considered.

Fig. 6 illustrates the relations between the average flow intensity $\overline{\lambda}$ or the average vehicle distance $\overline{\Delta\xi_0}$ and the first three cumulants of the random variable $w_{\mathrm{I}}^{P}(\xi)$ and $w_{\mathrm{II}}^{P}(\xi)$. One can see that the higher order of the cumulant, the smaller is the difference between the cumulant values.

In Fig. 7 the expected values and also the variances of the stochastic maximal displacement $w_{\mathrm{II}}^{P}(\xi)$ and $w_{\mathrm{II}}^{E}(\xi)$ are compared. As one can see, the Poisson distribution gives greater values of the expected value than the Erlang process within the whole range of the average flow intensity $\overline{\lambda}$, but in the case of the variance this relation changes within the range of higher values of the average flow intensity $\overline{\lambda}$ which leads to a conclusion that for higher intensity the stochastic beam deflection function gives values more scattered for the Erlang process than for Poisson.

The first case (I) can be used to estimate the reliability of the beam or plate with respect to fatigue, the second case (II) can be useful in the reliability problem of the beam or plate with respect to the maximum response.

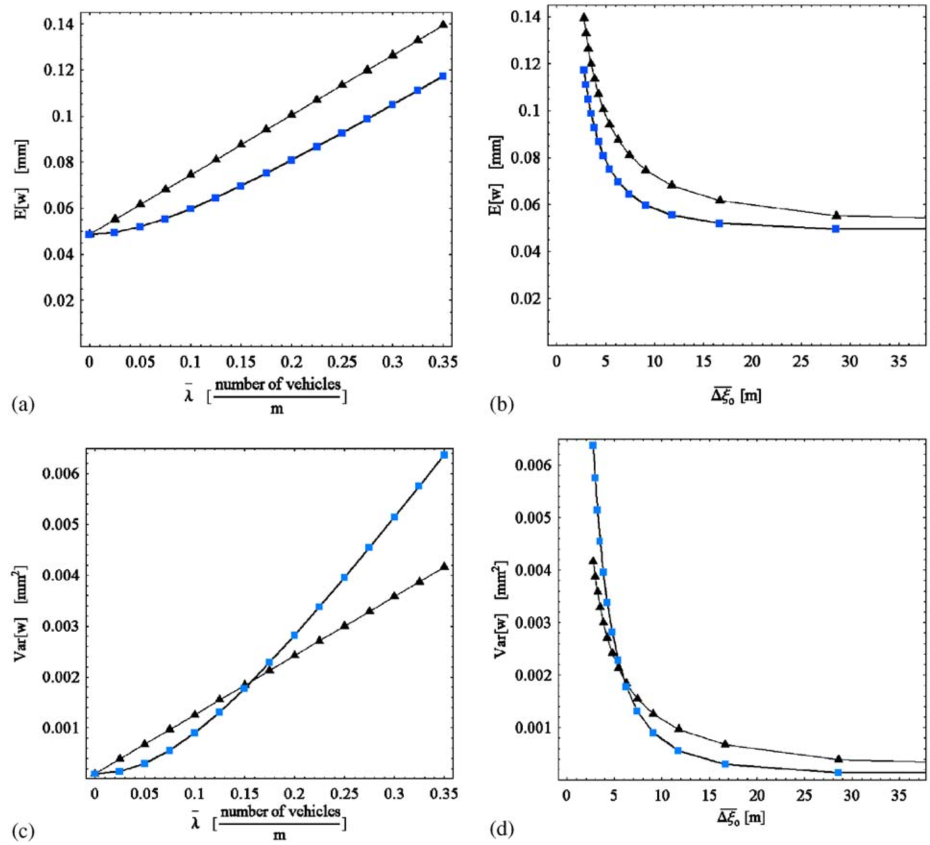

Fig. 7. The expected values and variances of a beam displacement in the second case (II) for the Poisson and the Erlang process in relation to the average flow intensity λ or the average vehicle distance $\Delta\xi_0$. (a), (b) expected value, (c), (d) variance. —▲—, values for Poisson process, —■—, values for the Erlang process.

One needs to notice that although performed method concerns in the paper only with the infinite beam and with the infinite plate simply supported at the sidelong edges, the presented approach is not confined only to the specified models of the girder. Moreover, the model of the subsoil may be arbitrary as well. Thus, all performed in the paper expressions for the random deflection and its probabilistic characteristics should be treated as universal and general solutions that are valid also for the finite models with arbitrary boundary-supporting conditions and also for the different models of the foundation. For such cases not the form of the expressions needs to be modified but only the dynamic influence function. Therefore eventual limits of the method are specified by the difficulties that may appear by determining formulas of the dynamic influence function for particular girder–subsoil models.

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Appendix A

Let us consider the steady-state vibrations of an infinite beam under a random series of forces A_i moving with a constant velocity v. Let the dynamic influence function $H(\xi, \xi_0)$ represent the steady-state displacement of the beam at the point ξ caused by a single force $A = 1$ moving with a velocity v and located at the point ξ_0. Our interest is to derive a formula for the nth order cumulant of the displacement in the particular time when one of the loads is situated in the place $\xi_0 = \xi$. In such a situation the deflection may be written in following form:

$$w(\xi) = A(\xi)H(\xi, \xi) + \int_{-\infty}^{\infty} A(\xi_0)H(\xi, \xi_0)\,\mathrm{d}N(\xi_0). \tag{A.1}$$

Let the functions $f_w(w)$ and $\phi_w(Q)$ be the probability density function and characteristic function of the random variable $w(\xi)$, respectively, i.e.:

$$f_w(w) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \exp[-\mathrm{i}Qw]\phi_w(Q)\,\mathrm{d}Q \tag{A.2}$$

and

$$\phi_w(Q) = E\{\exp[\mathrm{i}Qw]\} = \int_{-\infty}^{\infty} \exp[\mathrm{i}Qw]f_w(w)\mathrm{d}w, \tag{A.3}$$

where $w = w(\xi)$, $\mathrm{i} = \sqrt{-1}$.

After using Eq. (A.1) the expression in Eq. (A.3) takes the form:

$$\begin{aligned}\phi_w(Q) &= E\left\{\exp\left[\mathrm{i}Q\left(A(\xi)H(\xi,\xi) + \int_{-\infty}^{\infty} A(\xi_0)H(\xi,\xi_0)\,\mathrm{d}N(\xi_0)\right)\right]\right\} \\ &= E\{\exp[\mathrm{i}QA(\xi)H(\xi,\xi)]\}E\left\{\exp\left[\mathrm{i}Q\int_{-\infty}^{\infty} A(\xi_0)H(\xi,\xi_0)\,\mathrm{d}N(\xi_0)\right]\right\}.\end{aligned} \tag{A.4}$$

The integral in this expression can be regarded as a sum of many independent variables. After taking logarithms of both sides one obtains

$$\begin{aligned}\ln[\phi_w(Q)] &= \ln[E\{\exp(\mathrm{i}QA(\xi)H(\xi,\xi))\}] \\ &\quad + \int_{-\infty}^{\infty} \ln[E\{\exp(\mathrm{i}QA(\xi_0)H(\xi,\xi_0)\,\mathrm{d}N(\xi_0))\}] \\ &= \ln[\phi_A(QH(\xi,\xi))] \\ &\quad + \int_{-\infty}^{\infty} \ln[E\{\exp(\mathrm{i}QA(\xi_0)H(\xi,\xi_0)\,\mathrm{d}N(\xi_0))\}].\end{aligned} \tag{A.5}$$

Now let us deal with the integral in Eq. (A.5) and develop the exponential and the logarithmic function of ln(1+x)-type in Taylor's series. Considering the fact that for Poisson's distribution $E[\mathrm{d}N(\xi_0)] = \lambda(\xi_0)\,\mathrm{d}\xi_0$ and using the definition of the nth moment for the random variable $A(\xi_0)$, the expression from Eq. (A.5) can be transformed to the form:

$$\ln\left[\phi_w(Q)\right] = \ln\left[E\left\{\exp(\mathrm{i}QA(\xi)H(\xi,\xi))\right\}\right] + \int_{-\infty}^{\infty}\int_A \exp(\mathrm{i}QaH(\xi,\xi_0))f_A(a)\lambda\,\mathrm{d}a\,\mathrm{d}\xi_0 - \int_{-\infty}^{\infty}\lambda\,\mathrm{d}\xi. \tag{A.6}$$

Eq. (A.6) may be used to find the nth order cumulant of the random variables $w(\xi)$ according to the following formula:

$$\kappa_n[w(\xi)] = \frac{1}{\mathrm{i}^n}\frac{\mathrm{d}^n}{\mathrm{d}Q^n}\ln\left[\phi_w(Q)\right]\Big|_{Q=0}, \tag{A.7}$$

which gives

$$\kappa_n[w(\xi)] = \kappa_n[A]H^n(\xi,\xi) + E[A^n]\int_{-\infty}^{\infty} H^n(\xi,\xi_0)\lambda(\xi_0)\,\mathrm{d}\xi_0, \tag{A.8}$$

where κ_n[A] means the nth order cumulant of the random variables A_i.

Appendix B

Let us consider an infinite beam resting on a two-parameter foundation subjected to a singular point load moving with a constant velocity. Our interest is to derive an explicit expression for the dynamic influence function $H(\xi,\ \xi_0)$ where ξ denotes the cross-section in which the displacement is counted whereas ξ_0 is the load's location. The equation describing vibrations of the beam has the form:

$$\mathrm{EI}\frac{\partial^4 H(\xi,\xi_0)}{\partial\xi^4} + \left(mv^2 - k_1\right)\frac{\partial^2 H(\xi,\xi_0)}{\partial\xi^2} + k_0 H(\xi,\xi_0) = \delta(\xi-\xi_0), \tag{B.1}$$

where EI denotes the flexural rigidity of the beam, m denotes the mass per unit length of the beam, k_0 is the elastic stiffness of the foundation (Winkler's parameter), k_1 the shear stiffness of the foundation and v the velocity of moving load.

The Laplace transform is defined as follows:

$$\overline{f(s)} = \int_0^{\infty} \mathrm{e}^{-s\xi} f(\xi)\,\mathrm{d}\xi \tag{B.2}$$

and will be used in order to derive the unknown dynamic influence function.

From Eq. (B.1) the Laplace transform of the function $H(\xi,\ \xi_0)$ is obtained

$$\overline{H}(s,\xi_0) = \frac{\mathrm{e}^{-\xi_0 s} + s\left[s^2EI + mv^2 - k_1\right]H(0^+,\xi_0) + \left[s^2EI + mv^2 - k_1\right]H'(0^+,\xi_0) + sEI\,H''(0^+,\xi_0) + EI\,H'''(0^+,\xi_0)}{s^4EI + s^2[mv^2 - k_1] + k_0} \tag{B.3}$$

where

$$H^{(n)}(0^+,\xi_0) = \frac{\partial^n H(\xi,\xi_0)}{\partial\xi^n}\Big|_{\xi=0^+}.$$

Because the range of the variable ξ in the definition of the Laplace transform does not coincide with the range of a whole beam it is necessary to consider the deflection as a reaction to a superposition of the symmetric and antisymmetric load statically (also dynamically) equivalent to the original singular point load. This idea is illustrated in Fig. 8.

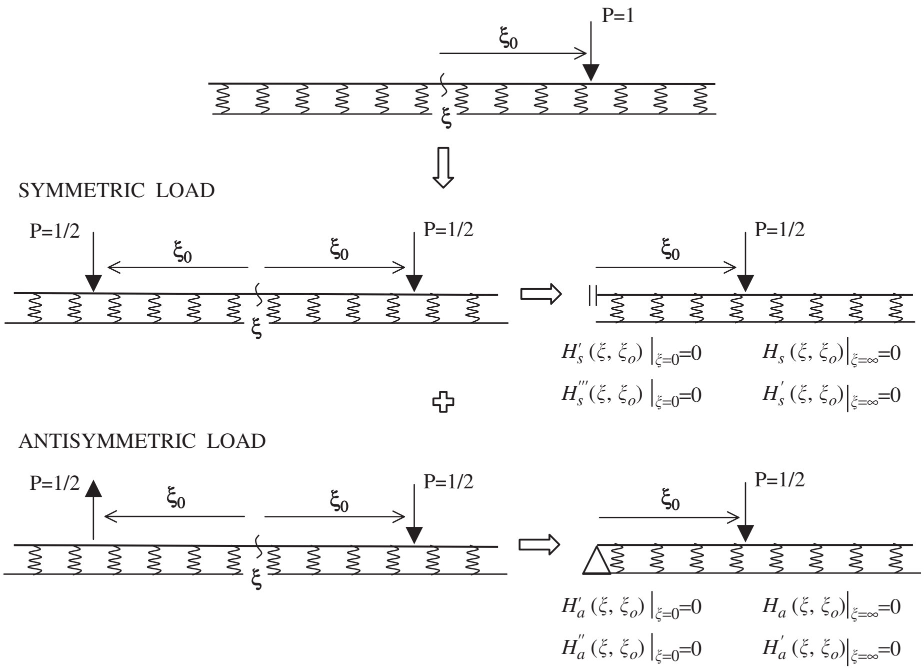

Fig. 8. The illustration of the idea for deriving the dynamic influence function $H(\xi, \xi_0)$.

Now using the Laplace transform given by Eq. (B.3) one has to solve the half-infinite beam for the symmetric and antisymmetric load taking into account the boundary condition shown in Fig. 8, which gives functions $H_s(\xi, \xi_0)$ and $H_a(\xi, \xi_0)$, respectively.

The final form of $H(\xi, \xi_0)$ is obtained by adding $H_s(\xi, \xi_0)$ and $H_a(\xi, \xi_0)$ for the range $\xi \geqslant 0$ and by subtracting $H_a(\xi, \xi_0)$ from $H_s(\xi, \xi_0)$ for the range $\xi < 0$, which leads to the formula given by Eq. (30).

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